

# SOME HOMOLOGICAL PROPERTIES OF $T$ -LAU PRODUCT ALGEBRA

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**ABSTRACT.** Let  $T$  be a homomorphism from a Banach algebra  $B$  to a Banach algebra  $A$ . The Cartesian product space  $A \times B$  with  $T$ -Lau multiplication and  $\ell^1$ -norm becomes a new Banach algebra  $A \times_T B$ . We investigate the notions such as approximate amenability, pseudo amenability,  $\phi$ -pseudo amenability,  $\phi$ -biflatness and  $\phi$ -biprojectivity for Banach algebra  $A \times_T B$ . We also present an example to show that approximate amenability of  $A$  and  $B$  is not stable for  $A \times_T B$ . Finally we characterize the double centralizer algebra of  $A \times_T B$  and present an application of this characterization.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $A$  and  $B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism. Then we consider  $A \times B$  with the following product

$$(a, b) \times_T (c, d) = (ac + T(b)c + aT(d), bd) \quad ((a, b), (c, d) \in A \times B).$$

The Cartesian product space  $A \times B$  with this product is denoted by  $A \times_T B$ . Let  $A$  and  $B$  be Banach algebras and let  $\|T\| \leq 1$ . Then we consider  $A \times_T B$  with the following norm

$$\|(a, b)\| = \|a\| + \|b\| \quad ((a, b) \in A \times_T B).$$

We note that  $A \times_T B$  is a Banach algebra with this norm.

Suppose that  $T : B \rightarrow A$  is an algebra homomorphism with  $\|T\| \leq 1$  and  $A$  is a commutative Banach algebra. Then Bhatt and Dabshi [2] have studied the properties, such as Gelfand space, Arens regularity and amenability of  $A \times_T B$ . Moreover suppose that  $A$  is unital with unit element  $e$  and  $\psi_0 : B \rightarrow \mathbb{C}$  is a multiplicative linear functional on  $B$ . If we define  $T : B \rightarrow A$  by  $T(b) = \psi_0(b)e$ , then the product  $\times_T$  coincides with the Lau product [8]. The group algebra  $L^1(G)$ , the measure algebra  $M(G)$ , the Fourier algebra  $A(G)$  of a locally compact group  $G$  and the Fourier-Stieltjes algebra of a topological group are the examples of Lau algebra [8]. Lau product was extended by Sangani Monfared for the general case [13]. Many basic properties of  $A \times_\theta B$  such as existence of a bounded approximate identity, spectrum, topological center, the ideal structure, biflatness and biprojectivity are investigated in [13] and [7].

Following [2], Abtahi *et al.* [1] for every Banach algebras  $A$  and  $B$  defined the Banach algebra  $A \times_T B$  equipped with algebra multiplication

$$(a, b) \times_T (c, d) = (ac + T(b)c + aT(d), bd) \quad ((a, b), (c, d) \in A \times B)$$

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and the norm  $\|(a, b)\| = \|a\| + \|b\|$ . They studied the homological properties of this Banach algebra such as biflatness, biprojectivity and existence of a approximate identity.

We recall some basic definition of the homological properties. A Banach algebra  $A$  is called biprojective if  $\pi_A : A \hat{\otimes} A \rightarrow A$  has a bounded right inverse which is an  $A$ -bimodule map. A Banach algebra  $A$  is called biflat if the adjoint map  $\pi_A^* : A^* \rightarrow (A \hat{\otimes} A)^*$  of  $\pi_A$  has a bounded left inverse which is an  $A$ -bimodule map. Here the product morphism  $\pi_A : A \hat{\otimes} A \rightarrow A$  for a Banach algebra  $A$  is defined by  $\pi_A(a \otimes b) = ab$ . It is clear that  $\pi_A$  is an  $A$ -bimodule map.

A Banach algebra  $A$  is called approximately amenable if for every  $A$ -bimodule  $X$ , any derivation  $D : A \rightarrow X^*$  is approximately inner. A Banach algebra  $A$  has an approximate identity if there is a net  $\{\eta_\alpha\} \subseteq A$  such that  $\lim \eta_\alpha a - a = \lim a \eta_\alpha - a = 0$  for all  $a \in A$ . A Banach algebra  $A$  has a weak approximate identity if there is a net  $\{\eta_\alpha\} \subseteq A$  such that  $\lim f(\eta_\alpha a - a) = \lim f(a \eta_\alpha - a) = 0$  for all  $a \in A$  and  $f \in A^*$ .

The notion of pseudo amenability of Banach algebras was introduced by Ghahramani and Zhang in [5]. A Banach algebra  $A$  is said to be pseudo amenable if there exists a net  $\{\rho_\alpha\} \subseteq A \hat{\otimes} A$ , such that  $a \rho_\alpha - \rho_\alpha a \rightarrow 0$  and  $\pi_A(\rho_\alpha) a \rightarrow a$  for all  $a \in A$  [5].

The notion of character pseudo amenability was introduced by Nasr-Isfahani and Nemati in [10]. Let  $A$  be a Banach algebra and let  $\phi \in \Delta(A)$ , where  $\Delta(A)$  is the character space of  $A$ . We say that  $A$  is  $\phi$ -pseudo amenable if it has a (right) $\phi$ -approximate diagonal, that is, there exists a not necessarily bounded net  $(m_\alpha) \subseteq A \hat{\otimes} A$  such that

$$\phi(\pi_A(m_\alpha)) \rightarrow 1 \quad \text{and} \quad \|a \cdot m_\alpha - \phi(a)m_\alpha\| \rightarrow 0$$

for all  $a \in A$  [10].

Let  $A$  be a Banach algebra and let  $\phi \in \Delta(A)$ . Then  $A$  is called  $\phi$ -biprojective if there exists a bounded  $A$ -bimodule morphism  $\rho : A \rightarrow A \hat{\otimes} A$  such that  $\phi \circ \pi_A \circ \rho(a) = \phi(a)$  for every  $a \in A$ . A Banach algebra  $A$  is called  $\phi$ -biflat if there exists a bounded  $A$ -bimodule morphism  $\rho_A : A \rightarrow (A \hat{\otimes} A)^{**}$  such that  $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$  for every  $a \in A$ , where  $\tilde{\phi}$  is a unique extension of  $\phi$  on  $A^{**}$  defined by  $\tilde{\phi}(F) = F(\phi)$  for every  $F \in A^{**}$ . It is clear that this extension remains to be a character on  $A^{**}$  [11].

The purpose of this paper is to determine some homological properties of  $A \times_T B$  for every Banach algebras  $A$  and  $B$ , such as approximate amenability, pseudo amenability,  $\varphi$ -pseudo amenability,  $\varphi$ -biflatness and  $\varphi$ -biprojectivity. After all we characterize the double centralizer algebra of  $A \times_T B$  and we will give an application of this characterization. We also present an example which show that  $A$  and  $B$  are approximately amenable but  $A \times_T B$  is not approximately amenable.

Recall that  $A^*$ , the dual of a Banach algebra  $A$ , is a Banach  $A$ -bimodule with the module operations defined by

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle, \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle, \quad (f \in A^*, a, b \in A).$$

Let  $A$  be a Banach algebra. The second dual  $A^{**}$  of  $A$  with Arens products  $\square$  and  $\diamond$  which are defined by  $\langle m \square n, f \rangle = \langle m, n \cdot f \rangle$ , where  $\langle n \cdot f, a \rangle = \langle n, f \cdot a \rangle$ , and similarly  $\langle m \diamond n, g \rangle = \langle n, f \cdot m \rangle$ , where  $\langle f \cdot m, a \rangle = \langle m, a \cdot f \rangle$  for every  $a \in A$ ,  $f \in A^*$  and  $m, n \in A^{**}$ , becomes a Banach algebra.

A Banach  $A$ -bimodule  $X$  is called neo-unital if for every  $x \in X$  there exist  $a, a' \in A$  and  $y, y' \in X$  such that  $ay = x = y'a'$ .

The dual space  $(A \times_T B)^*$  is identified with  $A^* \times B^*$  via

$$\langle (f, g), (a, b) \rangle = f(a) + g(b), \quad (a \in A, b \in B, f \in A^*, g \in B^*)$$

for more details see [9, Theorem 1.10.13]. Also the dual space  $(A \times_T B)^*$  is  $(A \times_T B)$ -bimodule with the module operations defined by

$$(a, b) \cdot (f, g) = (a \cdot f + T(b) \cdot f, (a \cdot f) \circ T + b \cdot g)$$

$$(f, g) \cdot (a, b) = (f \cdot a + f \cdot T(b), (f \cdot a) \circ T + g \cdot b).$$

Moreover  $A \times_T B$  is a Banach  $A$ -bimodule under the module actions  $a' \cdot (a, b) = (a', 0) \times_T (a, b)$  and  $(a, b) \cdot a' = (a, b) \times_T (a', 0)$ , for all  $a, a' \in A, b \in B$ . Similarly  $A \times_T B$  is a Banach  $B$ -bimodule.

## 2. $\phi$ -BIFLAT, $\phi$ -BIPROJECTIVE

In this section we investigate  $\phi$ -biflatness and  $\phi$ -biprojectivity of Banach algebra  $A \times_T B$

Recall that the character space of  $A \times_T B$  is determined by

$$\{(\phi, \phi \circ T) : \phi \in \Delta(A)\} \cup \{(0, \psi) : \psi \in \Delta(B)\}.$$

see [2, Theorem 2.1] for more details.

**Theorem 2.1.** *Let  $A, B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Then*

- (i) *If  $A \times_T B$  is  $(\phi, \phi \circ T)$ -biflat for  $\phi \in \Delta(A)$ , then  $A$  is  $\phi$ -biflat.*
- (ii) *If  $A \times_T B$  is  $(0, \psi)$ -biflat for  $\psi \in \Delta(B)$ , then  $B$  is  $\psi$ -biflat.*

*Proof.* (i) Since  $A \times_T B$  is  $(\phi, \phi \circ T)$ -biflat, there exists a  $A \times_T B$ -bimodule morphism  $\rho_{A \times_T B} : A \times_T B \rightarrow ((A \times_T B) \hat{\otimes} (A \times_T B))^{**}$  such that  $\pi_{A \times_T B}^{**} \circ \rho_{A \times_T B}(a, b)(\phi, \phi \circ T) = \phi(a) + \phi \circ T(b)$  for every  $(a, b) \in A \times_T B$ . We define  $\rho_A : A \rightarrow (A \hat{\otimes} A)^{**}$  by  $\rho_A = (r_A \otimes r_A)^{**} \circ \rho_{A \times_T B} \circ q_A$ , where  $r_A : A \times_T B \rightarrow A$  is defined by  $r_A((a, b)) = a + T(b)$  for every  $(a, b) \in A \times_T B$  and  $q_A : A \rightarrow A \times_T B$  is defined by  $q_A(a) = (a, 0)$  for every  $a \in A$ . We prove that for every  $a \in A$

$$(2.1) \quad \pi_A^{**} \circ \rho_A(a)(\phi) = \phi(a).$$

To calculate the left hand side of (2.1) we have

$$\begin{aligned} \pi_A^{**} \circ \rho_A(a)(\phi) &= \pi_A^{**} \circ (r_A \otimes r_A)^{**} \circ \rho_{A \times_T B} \circ q_A(a)(\phi) \\ &= \pi_A^{**} \circ (r_A \otimes r_A)^{**} \circ \rho_{A \times_T B}(a, 0)(\phi) \\ &= (r_A \otimes r_A)^{**} \circ \rho_{A \times_T B}(a, 0)(\pi_A^*(\phi)) \\ &= (r_A \otimes r_A)^{**} \circ \rho_{A \times_T B}(a, 0)(\phi \circ \pi_A) \\ &= \rho_{A \times_T B}(a, 0) \circ (r_A \otimes r_A)^*(\phi \circ \pi_A). \end{aligned} \tag{2.2}$$

To calculate the left hand side of (2.2) for every  $(x_1, y_1) \otimes (x_2, y_2) \in (A \times_T B) \hat{\otimes} (A \times_T B)$ , we have

$$\begin{aligned}
 (2.3) \quad & \rho_{A \times_T B}(a, 0) \circ (r_A \otimes r_A)^*(\phi \circ \pi_A)((x_1, y_1) \otimes (x_2, y_2)) \\
 &= \rho_{A \times_T B}(a, 0)(\phi \circ \pi_A(r_A(x_1, y_1) \otimes r_A(x_2, y_2))) \\
 &= \rho_{A \times_T B}(a, 0)(\phi \circ \pi_A(x_1 + T(y_1)) \otimes (x_2 + T(y_2))) \\
 &= \rho_{A \times_T B}(a, 0)(\phi(x_1 x_2) + \phi(x_1) \phi \circ T(y_2) \\
 &\quad + \phi \circ T(y_1) \phi(x_2) + \phi \circ T(y_1 y_2)).
 \end{aligned}$$

But for every  $(x_1, y_1) \otimes (x_2, y_2) \in (A \times_T B) \hat{\otimes} (A \times_T B)$ , we have

$$\begin{aligned}
 (2.4) \quad & \pi_{A \times_T B}^*(\phi, \phi \circ T)((x_1, y_1) \otimes (x_2, y_2)) = (\phi, \phi \circ T)(x_1 x_2 + T(y_1) x_2 + x_1 T(y_2), y_1 y_2) \\
 &= \phi(x_1 x_2) + \phi \circ T(y_1) \phi(x_2) + \phi(x_1) \phi \circ T(y_2) + \phi \circ T(y_1 y_2).
 \end{aligned}$$

Now If we replace left hand side of (2.4) in the last line of (2.3), we obtain

$$\begin{aligned}
 (2.5) \quad & \rho_{A \times_T B}(a, 0) \circ (r_A \otimes r_A)^*(\phi \circ \pi_A) = \rho_{A \times_T B}(a, 0)(\pi_{A \times_T B}^*(\phi, \phi \circ T)) \\
 &= \pi_{A \times_T B}^{**} \circ \rho_{A \times_T B}(a, 0)(\phi, \phi \circ T) \\
 &= \phi(a) + \phi \circ T(0) = \phi(a).
 \end{aligned}$$

Comparing (2.2) and (2.5) we see that  $\pi_A^{**} \circ \rho_A(a)(\phi) = \phi(a)$ . Moreover  $\rho_A$  is a  $A$ -bimodule map, therefore  $A$  is  $\phi$ -biflat.

(ii) Since  $A \times_T B$  is  $(0, \psi)$ -biflat, there exists a  $A \times_T B$ -bimodule morphism  $\rho_{A \times_T B} : A \times_T B \rightarrow ((A \times_T B) \hat{\otimes} (A \times_T B))^{**}$  such that for every  $(a, b) \in A \times_T B$  we have

$$\pi_{A \times_T B}^{**} \circ \rho_{A \times_T B}((a, b))(0, \psi) = (0, \psi)(a, b) = \psi(b).$$

Now we define  $\rho_B = (p_B \otimes p_B)^{**} \circ \rho_{A \times_T B} \circ q_B$ , where  $p_B : A \times_T B \rightarrow B$  is defined by  $p(a, b) = b$  and  $q_B : B \rightarrow A \times_T B$  is defined by  $q_B(b) = (0, b)$  for every  $(a, b) \in A \times_T B$ . So we have to show that

$$(2.6) \quad \pi_B^{**} \circ \rho_B(b)(\psi) = \psi(b) \quad (b \in B).$$

To calculate the left hand side of (2.6) we have

$$\begin{aligned}
 (2.7) \quad & \pi_B^{**} \circ \rho_B(b)(\psi) = \pi_B^{**} \circ (p_B \otimes p_B)^{**} \circ \rho_{A \times_T B}(0, b)(\psi) \\
 &= (p_B \otimes p_B)^{**} \circ \rho_{A \times_T B}(0, b)(\pi_B^*(\psi)) \\
 &= (p_B \otimes p_B)^{**} \circ \rho_{A \times_T B}(0, b)(\psi \circ \pi_B) \\
 &= \rho_{A \times_T B}(0, b) \circ (p_B \otimes p_B)^*(\psi \circ \pi_B).
 \end{aligned}$$

To calculate the left hand side of (2.7) for every  $(x_1, y_1) \otimes (x_2, y_2) \in (A \times_T B) \hat{\otimes} (A \times_T B)$ , we have

$$\begin{aligned}
 (2.8) \quad & \rho_{A \times_T B}(0, b) \circ (p_B \otimes p_B)^*(\psi \circ \pi_B)((x_1, y_1) \otimes (x_2, y_2)) \\
 &= \rho_{A \times_T B}(0, b)(\psi \circ \pi_B)(p_B \otimes p_B)((x_1, y_1) \otimes (x_2, y_2)) \\
 &= \rho_{A \times_T B}(0, b)(\psi \circ \pi_B)(y_1 \otimes y_2) \\
 &= \rho_{A \times_T B}(0, b)\psi(y_1 y_2).
 \end{aligned}$$

But  $\psi(y_1 y_2)$  in the above equation is obtained from

$$(2.9) \quad \pi_{A \times_T B}^*(0, \psi)((x_1, y_1) \otimes (x_2, y_2)) = (0, \psi)(x_1 x_2 + x_1 T(y_2) + T(y_1) x_2, y_1 y_2) = \psi(y_1 y_2).$$

Now if we replace (2.9) in (2.8) we obtain

$$\begin{aligned}
 (2.10) \quad \rho_{A \times_T B}(0, b) \circ (p_B \otimes p_B)^*(\psi \circ \pi_B) &= \rho_{A \times_T B}(0, b) \circ \pi_{A \times_T B}^*(0, \psi) \\
 &= \pi_{A \times_T B}^{**} \circ \rho_{A \times_T B}((0, b))(0, \psi) \\
 &= (0, \psi)(0, b) = \psi(b).
 \end{aligned}$$

Therefore (2.7) and (2.10) follow that  $\pi_B^{**} \circ \rho_B(b)(\psi) = \psi(b)$ . Since  $\rho_B$  is a  $B$ -bimodule map,  $B$  is  $\psi$ -biflat.  $\square$

**Theorem 2.2.** *Let  $A$  and  $B$  be Banach algebras, let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$  and let  $\phi \in \Delta(A)$  and  $\psi \in \Delta(B)$ . Then*

- (i)  $A \times_T B$  is  $(\phi, \phi \circ T)$ -biprojective if and only if  $A$  is  $\phi$ -biprojective.
- (ii)  $A \times_T B$  is  $(0, \psi)$ -biprojective if and only if  $B$  is  $\psi$ -biprojective.

*Proof.* (i) Let  $A \times_T B$  be  $(\phi, \phi \circ T)$ -biprojective. Then there exists a bounded  $A \times_T B$ -bimodule morphism  $\rho_{A \times_T B} : A \times_T B \rightarrow (A \times_T B) \hat{\otimes} (A \times_T B)$  such that

$$(2.11) \quad (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B} = (\phi, \phi \circ T).$$

To show that  $A$  is  $\phi$ -biprojective, we define  $\rho_A : A \rightarrow A \hat{\otimes} A$  by  $\rho_A = (r_A \otimes r_A) \circ \rho_{A \times_T B} \circ q_A$ , where  $q_A : A \rightarrow A \times_T B$  is defined by  $q_A(a) = (a, 0)$  for every  $a \in A$  and  $r_A : A \times_T B \rightarrow A$  is defined by  $r_A(a, b) = a + T(b)$ , for every  $(a, b) \in A \times_T B$ . Now we claim that for every  $a \in A$

$$(2.12) \quad \phi \circ \pi_A \circ \rho_A(a) = \phi(a).$$

For the left hand side of (2.12) we have

$$\begin{aligned}
 \phi \circ \pi_A \circ \rho_A(a) &= \phi \circ \pi_A \circ (r_A \otimes r_A) \circ \rho_{A \times_T B} \circ q_A(a) \\
 &= \phi \circ \pi_A \circ (r_A \otimes r_A) \circ \rho_{A \times_T B}(a, 0)
 \end{aligned}$$

Since  $\pi_A \circ (r_A \otimes r_A) = r_A \circ \pi_{A \times_T B}$ , we have

$$(2.13) \quad \phi \circ \pi_A \circ \rho_A(a) = \phi \circ r_A \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, 0).$$

On the other hand, for every  $(a, b) \in A \times_T B$  we have

$$\begin{aligned}
 (2.14) \quad (\phi \circ r_A)(a, b) &= \phi(r_A(a, b)) \\
 &= \phi(a + T(b)) \\
 &= \phi(a) + \phi \circ T(b) \\
 &= (\phi, \phi \circ T)(a, b).
 \end{aligned}$$

The equations (2.13) and (2.14) imply that

$$\phi \circ \pi_A \circ \rho_A(a) = (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, 0).$$

The equation (2.11) says that for every  $(a, b) \in A \times_T B$  we have  $(\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) = \phi(a) + \phi \circ T(b)$ . Hence we have

$$\begin{aligned} \phi \circ \pi_A \circ \rho_A(a) &= (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, 0) \\ &= \phi(a) + \phi \circ T(0) \\ &= \phi(a). \end{aligned}$$

Since  $\rho_A$  is a  $A$ -bimodule map, so  $A$  is  $\phi$ -biprojective.

Conversely, let  $A$  be  $\phi$ -biprojective. Then there exists a bounded  $A$ -bimodule morphism  $\rho : A \rightarrow A \hat{\otimes} A$  such that

$$(2.15) \quad \phi \circ \pi_A \circ \rho_A = \phi.$$

We define  $\rho_{A \times_T B} : A \times_T B \rightarrow (A \times_T B) \hat{\otimes} (A \times_T B)$  by  $\rho_{A \times_T B} = (q_A \otimes q_A) \circ \rho_A \circ r_A$ , where  $q_A : A \rightarrow (A \times_T B)$  is defined by  $q_A(a) = (a, 0)$  for every  $a \in A$  and  $r_A : A \times_T B \rightarrow A$  is defined by  $r_A(a, b) = a + T(b)$  for every  $(a, b) \in A \times_T B$ .

Now we show that for every  $(a, b) \in A \times_T B$

$$(2.16) \quad (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) = (\phi, \phi \circ T)(a, b) = \phi(a) + \phi \circ T(b).$$

To calculate the left hand side of (2.16) we have

$$\begin{aligned} (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) &= (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ (q_A \otimes q_A) \circ \rho_A \circ r_A(a, b) \\ &= (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ (q_A \otimes q_A) \circ \rho_A(a + T(b)). \end{aligned}$$

Since  $\pi_{A \times_T B} \circ (q_A \otimes q_A) = q_A \circ \pi_A$ , we have

$$(2.17) \quad \begin{aligned} (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) &= (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ (q_A \otimes q_A) \circ \rho_A(a + T(b)) \\ &= (\phi, \phi \circ T) \circ q_A \circ \pi_A \circ \rho_A(a + T(b)). \end{aligned}$$

In contrast, for every  $a \in A$  we have

$$(2.18) \quad \begin{aligned} (\phi, \phi \circ T) \circ q_A(a) &= (\phi, \phi \circ T)(a, 0) \\ &= \phi(a) + \phi \circ T(0) \\ &= \phi(a). \end{aligned}$$

The equation (2.17) and (2.18) follow that

$$\begin{aligned} (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) &= (\phi, \phi \circ T) \circ q_A \circ \pi_A \circ \rho_A(a + T(b)) \\ &= \phi \circ \pi_A \circ \rho_A(a + T(b)). \end{aligned}$$

The equation (2.15) says that for every  $a \in A$ , we have  $\phi \circ \pi_A \circ \rho_A(a) = \phi(a)$ .

Hence we have

$$\begin{aligned} (\phi, \phi \circ T) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) &= \phi \circ \pi_A \circ \rho_A(a + T(b)) \\ &= \phi(a + T(b)) \\ &= \phi(a) + \phi \circ T(b). \end{aligned}$$

We note that  $A$  is a Banach  $A \times_T B$ -bimodule with the following module actions

$$(a, b) \cdot c = (a + T(b))c, \quad c \cdot (a, b) = c(a + T(b)) \quad ((a, b) \in A \times_T B, c \in A).$$

Since  $\rho_{A \times_T B}$  is the composition of  $A \times_T B$ -bimodule maps, so  $A \times_T B$  is  $(\phi, \phi \circ T)$ -biprojective.

(ii) Let  $A \times_T B$  be  $(0, \psi)$ -biprojective. Then there exists a bounded  $A \times_T B$ -bimodule morphism  $\rho_{A \times_T B} : A \times_T B \rightarrow (A \times_T B) \hat{\otimes} (A \times_T B)$  such that

$$(2.19) \quad (0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B} = (0, \psi).$$

We define  $\rho_B : B \rightarrow B \hat{\otimes} B$  by  $\rho_B = (p_B \otimes p_B) \circ \rho_{A \times_T B} \circ q_B$ , where  $q_B : B \rightarrow A \times_T B$  is defined by  $q_B(b) = (0, b)$  for every  $b \in B$  and  $p_B : (A \times_T B) \rightarrow B$  is defined by  $p_B(a, b) = b$  for every  $(a, b) \in A \times_T B$ . We show that for every  $b \in B$

$$(2.20) \quad \psi \circ \pi_B \circ \rho_B(b) = \psi(b).$$

To calculate the left hand side of (2.20) we have

$$\begin{aligned} \psi \circ \pi_B \circ \rho_B(b) &= \psi \circ \pi_B \circ (p_B \otimes p_B) \circ \rho_{A \times_T B} \circ q_B(b) \\ &= \psi \circ \pi_B \circ (p_B \otimes p_B) \circ \rho_{A \times_T B}(0, b). \end{aligned}$$

Since  $\pi_B \circ (p_B \otimes p_B) = p_B \circ \pi_{A \times_T B}$ , therefore

$$(2.21) \quad \begin{aligned} \psi \circ \pi_B \circ \rho_B(b) &= \psi \circ \pi_B \circ (p_B \otimes p_B) \circ \rho_{A \times_T B}(0, b) \\ &= \psi \circ p_B \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(0, b). \end{aligned}$$

In contrast, for every  $(a, b) \in A \times_T B$  we have

$$(2.22) \quad \psi \circ p_B(a, b) = \psi(b) = (0, \psi)(a, b).$$

The equation (2.21) and (2.22) follow that for every  $(a, b) \in A \times_T B$

$$\begin{aligned} \psi \circ \pi_B \circ \rho_B(b) &= \psi \circ p_B \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(0, b) \\ &= (0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(0, b). \end{aligned}$$

The equation (2.19) says that for every  $(a, b) \in A \times_T B$

$$(0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) = \psi(b).$$

Hence, we have

$$\psi \circ \pi_B \circ \rho_B(b) = (0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(0, b) = \psi(b).$$

Since  $\rho_B$  is the composition of  $B$ -bimodule maps, so  $B$  is  $\psi$ -biprojective.

Conversely, let  $B$  be  $\psi$ -biprojective. Then there exists a bounded  $B$ -bimodule morphism  $\rho_B : B \rightarrow B \hat{\otimes} B$  such that

$$(2.23) \quad \psi \circ \pi_B \circ \rho_B = \psi.$$

We define  $\rho_{A \times_T B} : A \times_T B \rightarrow (A \times_T B) \hat{\otimes} (A \times_T B)$  by  $\rho_{A \times_T B} = (s_B \otimes s_B) \circ \rho_B \circ p_B$ , where  $s_B : B \rightarrow (A \times_T B)$  is defined by  $s_B(b) = (-T(b), b)$  for every  $b \in B$  and  $p_B : A \times_T B \rightarrow B$  is defined by  $p_B(a, b) = b$  for every  $(a, b) \in A \times_T B$ . We show that for every  $(a, b) \in A \times_T B$

$$(2.24) \quad (0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) = (0, \psi)(a, b) = \psi(b).$$

To calculate the left hand side of (2.24) we have

$$\begin{aligned} (0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) &= (0, \psi) \circ \pi_{A \times_T B} \circ (s_B \otimes s_B) \circ \rho_B \circ p_B(a, b) \\ &= (0, \psi) \circ \pi_{A \times_T B} \circ (s_B \otimes s_B) \circ \rho_B(b). \end{aligned}$$

Since  $\pi_{A \times_T B} \circ (s_B \otimes s_B) = s_B \circ \pi_B$ , we have

$$\begin{aligned} (2.25) \quad (0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) &= (0, \psi) \circ \pi_{A \times_T B} \circ (s_B \otimes s_B) \circ \rho_B(b) \\ &= (0, \psi) \circ s_B \circ \pi_B \circ \rho_B(b). \end{aligned}$$

In contrast, for every  $b \in B$  we have

$$(2.26) \quad (0, \psi) \circ s_B(b) = (0, \psi)(-T(b), b) = \psi(b).$$

The equations (2.25) and (2.26) follow that for every  $(a, b) \in A \times_T B$

$$\begin{aligned} (0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) &= (0, \psi) \circ s_B \circ \pi_B \circ \rho_B(b) \\ &= \psi \circ \pi_B \circ \rho_B(b). \end{aligned}$$

The equation (2.23) says that for every  $b \in B$ ,  $\psi \circ \pi_B \circ \rho_B(b) = \psi(b)$ . Hence we have

$$(0, \psi) \circ \pi_{A \times_T B} \circ \rho_{A \times_T B}(a, b) = \psi \circ \pi_B \circ \rho_B(b) = \psi(b).$$

We note that  $B$  is a Banach  $A \times_T B$ -bimodule by the module actions

$$(a, b) \cdot d = bd, \quad d \cdot (a, b) = db \quad ((a, b) \in A \times_T B, d \in B).$$

Hence  $\rho_B$  is  $A \times_T B$ -bimodule map. Since  $\rho_{A \times_T B}$  is the composition of  $A \times_T B$ -bimodule maps, so  $A \times_T B$  is  $(0, \psi)$ -biprojective.  $\square$

### 3. APPROXIMATE AMENABILITY AND PSEUDO AMENABILITY

In this section we investigate the approximate amenability and pseudo amenability of  $A \times_T B$ . We also provide a necessary and sufficient conditions for  $(\phi, \phi \circ T)$ -pseudo amenability and  $(0, \psi)$ -pseudo amenability of  $A \times_T B$ . Next lemma is similar to [1, Proposition 3.2], which we omit the proof.

**Lemma 3.1.** *Let  $A$  and  $B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Then*

- (i) *If  $\{(e_\alpha, \eta_\alpha)\}$  is (bounded) weakly approximate identity for  $A \times_T B$ , then  $\{e_\alpha + T(\eta_\alpha)\}$  and  $\{\eta_\alpha\}$  are (bounded) weakly approximate identities for  $A$  and  $B$ , respectively.*
- (ii) *If  $\{e_\alpha\}$  and  $\{\eta_\beta\}$  are (bounded) weakly approximate identities for  $A$  and  $B$ , respectively, then  $\{(e_\alpha - T(\eta_\beta), \eta_\beta)\}$  is (bounded) weakly approximate identity for  $A \times_T B$ .*

We use an analogue version of the method used in the [2, Theorem 4.1] to prove the next theorem.

**Theorem 3.2.** *Let  $A$  and  $B$  be Banach algebras and let  $A \times_T B$  be approximately amenable for an algebra homomorphism  $T : B \rightarrow A$  with  $\|T\| \leq 1$ . Then  $A$  and  $B$  are approximately amenable.*



*Proof.* Suppose that  $X$  is a Banach  $A$ -bimodule and  $d : A \rightarrow X^*$  is a bounded derivation. Simply we consider this derivation as a new derivation  $d : A \times \{0\} \rightarrow X^* \times \{0\}$ . Moreover,  $X \times \{0\}$  is a Banach  $A \times_T B$ -bimodule. Let  $\phi : A \times_T B \rightarrow A \times \{0\}$  be a map defined by

$$\phi(a, b) = (a + T(b), 0) \quad ((a, b) \in A \times_T B).$$

Now we take  $D : A \times_T B \rightarrow X^* \times \{0\}$  defined by  $D = d \circ \phi$ . Then  $D$  is a bounded derivation on  $A \times_T B$ . Since  $A \times_T B$  is approximately amenable, there exists a net  $(y_\alpha) \subseteq X^*$  such that

$$D(a, b) = \lim((a, b) \cdot (y_\alpha, 0) - (y_\alpha, 0) \cdot (a, b)) \quad ((a, b) \in A \times_T B).$$

Therefore

$$d(a) = d(a, 0) = D(a, 0) = \lim((a, 0) \cdot (y_\alpha, 0) - (y_\alpha, 0) \cdot (a, 0)) = \lim(a \cdot y_\alpha - y_\alpha \cdot a).$$

which means that  $A$  is approximately amenable.

Similarly, take  $\phi : A \times_T B \rightarrow \{0\} \times B$  defined by  $\phi(a, b) = (0, b)$ , it follows that  $B$  is approximately amenable.  $\square$

In the following example we show that the converse of the previous theorem is not valid in general case. In fact we show that if  $A$  and  $B$  are approximately amenable, then  $A \times_T B$  is not necessarily approximately amenable for an algebra homomorphism  $T : B \rightarrow A$  with  $\|T\| \leq 1$ .

**Example 3.3.** Let  $l^1$  denote the well-known space of complex sequences and let  $K(l^1)$  be the space of all compact operators on  $l^1$ . It is well known that the Banach algebra  $K(l^1)$  is amenable. We renorm  $K(l^1)$  with the family of equivalent norm  $\|\cdot\|^k$  such that its bounded left approximate identity will be the constant 1 and its bounded right approximate identity will be  $k + 1$ .

Now if we consider  $c_0$ -direct-sum  $A = \oplus_{k=1}^{\infty} (K(l^1), \|\cdot\|^k)$ , then  $A$  has a bounded left approximate identity but no bounded right approximate identity, see [4, page 3931] for more details. Now by choosing  $T = 0$  we have  $A \oplus A^{op} = A \times_T A^{op}$ . Note that  $A$  and  $A^{op}$ , the opposite algebra, are boundedly approximately amenable [4, Theorem 3.1], but since  $A \oplus A^{op}$  has no bounded approximate identity, it is not approximately amenable [4, Theorem 4.1].

**Theorem 3.4.** *Let  $A$  and  $B$  be Banach algebras and let  $A \times_T B$  be pseudo amenable for an algebra homomorphism  $T : B \rightarrow A$  with  $\|T\| \leq 1$ . Then  $A$  and  $B$  are pseudo amenable.*

*Proof.* We define  $\psi : A \times_T B \rightarrow B$  by  $\psi(a, b) = b$ , for every  $(a, b) \in A \times_T B$ . This map is a continuous epimorphism from  $A \times_T B$  onto  $B$ , hence  $B$  is pseudo amenable [5, Proposition 2.2]. Similarly, the map  $\psi : A \times_T B \rightarrow A$  defined by  $\psi(a, b) = a + T(b)$  is a continuous epimorphism from  $A \times_T B$  onto  $A$ . Hence  $A$  is pseudo amenable [5, Proposition 2.2].  $\square$

The concept of (right)  $\phi$ -pseudo amenability of Banach algebra  $A$  is equivalent with the existence of an approximate  $\phi$ -mean. (An approximate  $\phi$ -mean is a not necessarily bounded net  $\{a_\alpha\} \subseteq A$  such that

$$\phi(a_\alpha) \rightarrow 1 \quad \text{and} \quad \|aa_\alpha - \phi(a)a_\alpha\| \rightarrow 0$$

for all  $a \in A$ ).

**Theorem 3.5.** *Let  $A$  and  $B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Then*

- (i)  $A \times_T B$  is  $(\phi, \phi \circ T)$ -pseudo amenable if and only if  $A$  is  $\phi$ -pseudo amenable,
- (ii)  $A \times_T B$  is  $(0, \psi)$ -pseudo amenable if and only if  $B$  is  $\psi$ -pseudo amenable,

where  $\phi \in \Delta(A)$  and  $\psi \in \Delta(B)$ .

*Proof.* (i) Let  $A \times_T B$  be  $(\phi, \phi \circ T)$ -pseudo amenable. Then there exists a net  $\{(a_\alpha, b_\alpha)\} \subseteq A \times_T B$  such that

$$(3.1) \quad (\phi, \phi \circ T)(a_\alpha, b_\alpha) = \phi(a_\alpha) + \phi \circ T(b_\alpha) \rightarrow 1$$

and

$$(3.2) \quad \|((a, b)(a_\alpha, b_\alpha) - (\phi, \phi \circ T)((a, b)(a_\alpha, b_\alpha)))\| \rightarrow 0 \quad ((a, b) \in A \times_T B).$$

We show that  $\{a_\alpha + T(b_\alpha)\}$  is an approximate  $\phi$ -mean for  $A$ . By equation (3.2) we have

$$(3.3) \quad \begin{aligned} & \|((a, b)(a_\alpha, b_\alpha) - (\phi, \phi \circ T)((a, b)(a_\alpha, b_\alpha)))\| \\ &= \|(aa_\alpha + aT(b_\alpha) + T(b)a_\alpha - (\phi(a) + \phi \circ T(b))(a_\alpha, b_\alpha))\| \rightarrow 0. \end{aligned}$$

Substituting  $b = 0$  in the equation (3.3), we have

$$\|(aa_\alpha + aT(b_\alpha), 0) - (\phi(a)a_\alpha, \phi(a)b_\alpha)\| = \|(aa_\alpha + aT(b_\alpha) - \phi(a)a_\alpha, -\phi(a)b_\alpha)\| \rightarrow 0.$$

Hence we have

$$\|aa_\alpha + aT(b_\alpha) - \phi(a)a_\alpha\| \rightarrow 0$$

and

$$\|\phi(a)b_\alpha\| \rightarrow 0.$$

In contrast, we have

$$(3.4) \quad \begin{aligned} \|aa_\alpha + aT(b_\alpha) - \phi(a)a_\alpha - \phi(a)T(b_\alpha)\| &\leq \|aa_\alpha + aT(b_\alpha) - \phi(a)a_\alpha\| + \|\phi(a)T(b_\alpha)\| \\ &\leq \|aa_\alpha + aT(b_\alpha) - \phi(a)a_\alpha\| + \|\phi(a)b_\alpha\| \rightarrow 0. \end{aligned}$$

The equations (3.4) and (3.1) imply that  $\{a_\alpha + T(b_\alpha)\}$  is an approximate  $\phi$ -mean for  $A$ .

Conversely, let  $A$  be  $\phi$ -pseudo amenable. Then there exists a net  $\{a_\alpha\} \subseteq A$  such that  $\phi(a_\alpha) \rightarrow 1$  and  $\|aa_\alpha - \phi(a)a_\alpha\| \rightarrow 0$  for all  $a \in A$ . We show that  $\{(a_\alpha, 0)\}$  is an approximate  $(\phi, \phi \circ T)$ -mean for  $A \times_T B$ . We have  $(\phi, \phi \circ T)(a_\alpha, 0) = \phi(a_\alpha) \rightarrow 1$ . On the other hand for every  $(a, b) \in A \times_T B$  we have

$$\begin{aligned} \|(a, b)(a_\alpha, 0) - (\phi(a) + \phi \circ T(b))(a_\alpha, 0)\| &= \|(aa_\alpha + T(b)a_\alpha, 0) - (\phi(a)a_\alpha + \phi \circ T(b)a_\alpha, 0)\| \\ &= \|(aa_\alpha - \phi(a)a_\alpha + T(b)a_\alpha - \phi \circ T(b)a_\alpha, 0)\| \\ &\leq \|aa_\alpha - \phi(a)a_\alpha\| + \|T(b)a_\alpha - \phi \circ T(b)a_\alpha\| \rightarrow 0. \end{aligned}$$

Hence  $\{(a_\alpha, 0)\} \subseteq A \times_T B$  is an approximate  $(\phi, \phi \circ T)$ -mean for  $A \times_T B$ .

(ii) Suppose that  $A \times_T B$  is  $(0, \psi)$ -pseudo amenable. Then there exists a net  $\{(a_\alpha, b_\alpha)\} \subseteq A \times_T B$  such that for every  $(a, b) \in A \times_T B$  we have

$$(3.5) \quad (0, \psi)(a_\alpha, b_\alpha) = \psi(b_\alpha) \rightarrow 1$$

and

$$(3.6) \quad \|(a, b)(a_\alpha, b_\alpha) - (0, \psi)((a, b))(a_\alpha, b_\alpha)\| \rightarrow 0.$$

By equation (3.6) we have

$$\begin{aligned} \|(a, b)(a_\alpha, b_\alpha) - (0, \psi)((a, b))(a_\alpha, b_\alpha)\| &= \|(aa_\alpha + T(b)a_\alpha + aT(b_\alpha), bb_\alpha) - (\psi(b)a_\alpha, \psi(b)b_\alpha)\| \\ &= \|(aa_\alpha + T(b)a_\alpha + aT(b_\alpha) - \psi(b)a_\alpha, bb_\alpha - \psi(b)b_\alpha)\| \rightarrow 0. \end{aligned}$$

Hence we have

$$\|aa_\alpha + T(b)a_\alpha + aT(b_\alpha) - \psi(b)a_\alpha\| \rightarrow 0$$

and

$$(3.7) \quad \|bb_\alpha - \psi(b)b_\alpha\| \rightarrow 0.$$

The equations (3.5) and (3.7) follow that  $\{b_\alpha\} \subseteq B$  is an approximate  $\psi$ -mean for  $B$ .

Conversely, suppose that  $B$  is  $\psi$ -pseudo amenable. Then there exists a net  $\{b_\alpha\} \subseteq B$  such that for every  $b \in B$

$$(3.8) \quad \psi(b_\alpha) \rightarrow 1 \quad \text{and} \quad \|bb_\alpha - \psi(b)b_\alpha\| \rightarrow 0.$$

We show that the net  $(-T(b_\alpha), b_\alpha) \subseteq A \times_T B$  is an approximate  $(0, \psi)$ -mean for  $A \times_T B$ . Using (3.8) yields

$$(3.9) \quad (0, \psi)(-T(b_\alpha), b_\alpha) = \psi(b_\alpha) \rightarrow 1.$$

Moreover, for every  $(a, b) \in A \times_T B$  we have

$$\begin{aligned} \|(a, b)(-T(b_\alpha), b_\alpha) - (0, \psi)((a, b))(-T(b_\alpha), b_\alpha)\| &= \|(-aT(b_\alpha) + aT(b_\alpha) \\ &\quad - T(b)T(b_\alpha), bb_\alpha) - (\psi(b)(-T(b_\alpha), b_\alpha))\| \\ &= \|(-T(b)T(b_\alpha), bb_\alpha) - (-\psi(b)T(b_\alpha), \psi(b)b_\alpha)\| \\ &= \|(-T(b)T(b_\alpha) + \psi(b)T(b_\alpha), bb_\alpha - \psi(b)b_\alpha)\| \\ &= \|-T(b)T(b_\alpha) - \psi(b)T(b_\alpha)\| + \|bb_\alpha - \psi(b)b_\alpha\|. \end{aligned}$$

The equation (3.8) follows that

$$T(bb_\alpha) - \psi(b)T(b_\alpha) \rightarrow 0.$$

Hence for every  $(a, b) \in A \times_T B$  we have

$$(3.10) \quad \|(a, b)(-T(b_\alpha), b_\alpha) - (0, \psi)((a, b))(-T(b_\alpha), b_\alpha)\| \rightarrow 0.$$

The equations (3.10) and (3.9) follow that  $\{(-T(b_\alpha), b_\alpha)\}$  is an approximate  $(0, \psi)$ -mean for  $A \times_T B$ .  $\square$

## 4. DOUBLE CENTRALIZER ALGEBRA

In this section we characterize double centralizer algebra  $M(A \times_T B)$  of  $A \times_T B$  and as an application we obtain a sufficient condition for approximate amenability of  $A \times_T B$ .

If  $A$  is a Banach algebra, then the idealizer  $Q(A)$  of  $A$  in  $(A^{**}, \square)$  is defined by

$$Q(A) = \{f_1 \in A^{**} : x \square f_1 \text{ and } f_1 \square x \in A \text{ for every } x \in A\},$$

see [12] for more details.

**Proposition 4.1.** *Let  $A$  and  $B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Then  $Q(A \times_T B) \cong Q(A) \times Q(B)$ .*

*Proof.* By a similar method as in [2, Theorem 3.1], we have  $(A \times_T B)^{**} \cong A^{**} \times_{T^{**}} B^{**}$ . We define a map  $\phi : Q(A \times_T B) \rightarrow Q(A) \times Q(B)$  by  $\phi(f_1, f_2) = (f_1 + T^{**}(f_2), f_2)$  for every  $(f_1, f_2) \in Q(A \times_T B) \subseteq (A \times_T B)^{**} \cong A^{**} \times_{T^{**}} B^{**}$ . Since  $(f_1, f_2) \in Q(A \times_T B)$  it follows that for every  $(a, b) \in A \times_T B$ , we have  $(f_1, f_2) \square (a, b) \in A \times_T B$ . With a similar calculation as in [2, Theorem 3.1], we obtain that

$$(f_1, f_2) \square (a, b) = (f_1 \square a + T^{**}(f_2) \square a + f_1 \square T^{**}(b), f_2 \square b).$$

Hence

$$(4.1) \quad (f_1 \square a + T^{**}(f_2) \square a + f_1 \square T^{**}(b), f_2 \square b) \in A \times_T B.$$

Substituting  $b = 0$  in (4.1) follows that  $f_1 \square a + T^{**}(f_2) \square a \in A$ , for all  $a \in A$ . Similarly  $a \square f_1 + a \square T^{**}(f_2) \in A$ , for every  $a \in A$ . Hence  $f_1 + T^{**}(f_2) \in Q(A)$ . Clearly, equation (4.1) shows that  $f_2 \square b$  and similarly  $b \square f_2 \in B$  for all  $b \in B$ . Therefore  $f_2 \in Q(B)$ . Hence the map  $\phi$  is well-defined. Clearly the map  $\phi$  is an isomorphism from  $Q(A \times_T B)$  onto  $Q(A) \times Q(B)$ . This completes the proof.  $\square$

Recall that  $(A \times_T B)^* \cong (A^* \times B^*)$  [9, Theorem 1.10.13] and with a similar argument as in [2, theorem 3.1], we have  $(A \times_T B)^{**} \cong A^{**} \times_{T^{**}} B^{**}$ . Hence we have  $(A \times_T B)^{***} \cong A^{***} \times B^{***}$ .

**Theorem 4.2.** *Let  $A$  and  $B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . If  $A \times_T B$  has weakly approximate identity  $\{(e_\alpha, \eta_\alpha)\}$  such that  $w - \lim(f, g) \square (e_\alpha, \eta_\alpha) = (f, g)$  for all  $(f, g) \in (A \times_T B)^{**}$ , then  $M(A \times_T B) \cong M(A) \times M(B)$ .*

*Proof.* Suppose that the hypothesis holds. Then using [12, Corollary 2], it implies that  $Q(A \times_T B) \cong M(A \times_T B)$ . Since  $\{(e_\alpha, \eta_\alpha)\}$  is weakly approximate identity for  $A \times_T B$ , by Lemma 3.1,  $\{e_\alpha + T(\eta_\alpha)\}$  and  $\{\eta_\alpha\}$  are weakly approximate identities for  $A$  and  $B$ , respectively.

Since  $w - \lim(f, g) \square (e_\alpha, \eta_\alpha) = (f, g)$  for all  $(f, g) \in (A \times_T B)^{**}$ , it follows that  $(f^{***}, g^{***})((f, g) \square (e_\alpha, \eta_\alpha)) \rightarrow (f^{***}, g^{***})(f, g)$  for every  $(f^{***}, g^{***}) \in (A \times_T B)^{***}$ . Hence for every  $(f^{***}, g^{***}) \in A^{***} \times B^{***}$  and  $(f, g) \in A^{**} \times_{T^{**}} B^{**}$  we have

$$(4.2) \quad (f^{***}, g^{***})(f \square e_\alpha + f \square T(\eta_\alpha) + T(g) \square e_\alpha, g \square \eta_\alpha) \rightarrow f^{***}(f) + g^{***}(g).$$

Substituting  $g = 0$  in (4.2), we obtain  $f^{***}(f \square e_\alpha + f \square T(\eta_\alpha)) \rightarrow f^{***}(f)$  for every  $f^{***} \in A^{***}$  and  $f \in A^{**}$ . It follows that  $w - \lim f \square (e_\alpha + T(\eta_\alpha)) = f$  for every  $f \in A^{**}$ . Using this fact and since  $\{e_\alpha + T(\eta_\alpha)\}$  is weakly approximate identity for  $A$ , by [12, Corollary 2] we have  $M(A) \cong Q(A)$ .

Substituting  $f^{***} = 0$  in (4.2), it follows that  $g^{***}(g \square \eta_\alpha) \rightarrow g^{***}(g)$  for every  $g^{***} \in B^{***}$  and  $g \in B^{**}$ . Hence we have  $w - \lim g \square \eta_\alpha = g$  for every  $g \in B^{**}$ . Again using this fact and since  $\{\eta_\alpha\}$  is weakly approximate identity for  $B$ , by [12, Corollary 2], we have  $M(B) \cong Q(B)$ . Combining these facts and Proposition 4.1 yields

$$M(A \times_T B) \cong Q(A \times_T B) \cong Q(A) \times Q(B) \cong M(A) \times M(B).$$

□

**Theorem 4.3.** *Let  $A$  and  $B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . If  $A \times_T B$  has weakly bounded approximate identity, then  $M(A \times_T B) \cong M(A) \times M(B)$ .*

*Proof.* Since  $A \times_T B$  has weakly bounded approximate identity, [12, Theorem 2] shows that the map  $\phi : Q(A \times_T B) \rightarrow M(A \times_T B)$  defined by  $\phi(f_1, f_2) = (L_{(f_1, f_2)}, R_{(f_1, f_2)})$  for every  $(f_1, f_2) \in Q(A \times_T B)$  is onto. Set

$$K_{A \times_T B} = \{(f_1, f_2) \in A^{**} \times_{T^{**}} B^{**} : (g_1, g_2) \square (f_1, f_2) = 0, \quad \forall (g_1, g_2) \in (A^{**} \times_{T^{**}} B^{**})\}.$$

By [12, Theorem 1]  $\ker \phi = K_{A \times_T B} \cap Q(A \times_T B)$ . This implies that

$$(4.3) \quad \frac{Q(A \times_T B)}{K_{A \times_T B} \cap Q(A \times_T B)} \cong M(A \times_T B),$$

by a similar argument, we have

$$(4.4) \quad \frac{Q(A)}{K_A \cap Q(A)} \cong M(A)$$

and

$$(4.5) \quad \frac{Q(B)}{K_B \cap Q(B)} \cong M(B),$$

where  $\cong$  denotes the algebra isomorphism.

Now we define  $\psi : K_{A \times_T B} \rightarrow K_A \times K_B$  by  $\psi(f_1, f_2) = (f_1 + T^{**}(f_2), f_2)$ . Clearly  $\psi$  is an algebra isomorphism. Hence we have  $K_{A \times_T B} \cong K_A \times K_B$  and by Proposition 4.1 we have

$$(4.6) \quad Q(A \times_T B) \cong Q(A) \times Q(B).$$

Therefore

$$(4.7) \quad \begin{aligned} K_{A \times_T B} \cap Q(A \times_T B) &\cong (K_A \times K_B) \cap (Q(A) \times Q(B)) \\ &= (K_A \cap Q(A)) \times (K_B \cap Q(B)). \end{aligned}$$

The equations (4.6) and (4.7) imply that

$$(4.8) \quad \begin{aligned} \frac{Q(A \times_T B)}{K_{A \times_T B} \cap Q(A \times_T B)} &\cong \frac{Q(A) \times Q(B)}{(K_A \cap Q(A)) \times (K_B \cap Q(B))} \\ &\cong \frac{Q(A)}{K_A \cap Q(A)} \times \frac{Q(B)}{K_B \cap Q(B)}. \end{aligned}$$

Combining (4.3), (4.4) and (4.5) yields  $M(A \times_T B) \cong M(A) \times M(B)$ . □

Now as an application of the previous theorem we give a version of inverse of Theorem 3.2, which is not valid in general case.

**Theorem 4.4.** *Let  $A$  and  $B$  be Banach algebras and let  $T : B \rightarrow A$  be an algebra homomorphism with  $\|T\| \leq 1$ . Suppose that  $A$  is approximately amenable,  $A$  has a bounded approximate identity and  $B$  is amenable. Then  $A \times_T B$  is approximately amenable.*

*Proof.* Let  $\{e_\alpha; \alpha \in I\}$  and  $\{\eta_\beta; \beta \in J\}$  be bounded approximate identities for  $A$  and  $B$ , respectively. By Lemma 3.1,  $\{(e_\alpha - T(\eta_\beta), \eta_\beta)\}$  is a bounded approximate identity for  $A \times_T B$ . Hence by [6, Proposition 1.8]  $A \times_T B$  is approximate amenable if and only if every bounded derivation from  $A \times_T B$  into  $X^*$  is approximate inner, where  $X$  is a neo-unital Banach  $A \times_T B$ -bimodule.

Since  $M(A \times_T B) \cong M(A) \times M(B)$ , we can define two maps  $a \mapsto a \times \{1\}$  and  $b \mapsto \{1\} \times b$ , which are homomorphisms of  $A$  and  $B$  into two commuting subsets of  $M(A \times_T B)$ , which are denoted by  $A \times \{1\}$  and  $\{1\} \times B$ , respectively. With a similar argument as in [6, Proposition 1.9] we see that  $X$  is a Banach  $M(A \times_T B)$ -bimodule, so it is an Banach  $A$ -bimodule and a Banach  $B$ -bimodule. Let  $D \in \mathcal{Z}^1(A \times_T B, X^*)$ . Then  $D$  extends to an element of  $\mathcal{Z}^1(M(A \times_T B), X^*)$  still denoted by  $D$  and this gives an element  $D_1$  of  $\mathcal{Z}^1(B, X^*)$  by restriction to  $\{1\} \times B$ . Since  $B$  is amenable, there exists  $y_0 \in X^*$  such that  $D_1(b) = ad_{y_0}(b) = y_0 \cdot b - b \cdot y_0$ , for every  $b \in B$ . Now let  $\tilde{D} = D - ad_{y_0} \in \mathcal{Z}^1(M(A \times_T B), X^*)$ , we have  $\tilde{D}(1, b) = 0$  for every  $b \in B$ , hence for every  $a \in A$  and  $b \in B$  we have

$$(1, b) \cdot \tilde{D}(a, 1) = \tilde{D}(a, b) = \tilde{D}(a, 1) \cdot (1, b),$$

which shows that the range of  $\tilde{D}|_{A \times \{1\}}$  is in  $\mathcal{Z}^0(\{1\} \times B, X^*) = \{x^* \in X^* : x^* \cdot (1, b) = (1, b) \cdot x^* \quad (b \in B)\}$ . Since  $A$  is approximately amenable,  $A$  is approximately contractible [3, Theorem 2.1]. Hence there exists  $\{z_\alpha\} \subseteq \mathcal{Z}^0(\{1\} \times B, X^*)$  such that

$$\tilde{D}(a, 1) = \lim z_\alpha \cdot (a, 1) - (a, 1) \cdot z_\alpha.$$

Since  $\{z_\alpha\} \subseteq \mathcal{Z}^0(\{1\} \times B, X^*)$ , we have  $\tilde{D} = 0$  on  $\{1\} \times B$ . Now we define derivation  $\tilde{\tilde{D}} \in \mathcal{Z}^1(M(A \times_T B), X^*)$  by  $\tilde{\tilde{D}}(a, b) = \tilde{D}(a, b) - (\lim z_\alpha \cdot (a, b) - (a, b) \cdot z_\alpha)$  for every  $(a, b) \in M(A \times_T B) \cong M(A) \times M(B)$ . we have  $\tilde{\tilde{D}} = 0$  on  $(A \times \{1\}) \cup (\{1\} \times B)$ . Since  $A \times_T B \hookrightarrow M(A \times_T B) \cong M(A) \times M(B)$ , for every  $(a, b) \in A \times_T B$

$$\tilde{\tilde{D}}(a, b) = \tilde{\tilde{D}}((a, 1)(1, b)) = (a, 1) \cdot \tilde{\tilde{D}}(1, b) + \tilde{\tilde{D}}(a, 1) \cdot (1, b) = 0,$$

that is,  $\tilde{D}(a, b) - (\lim z_\alpha \cdot (a, b) - (a, b) \cdot z_\alpha) = 0$  for every  $(a, b) \in A \times_T B$ .

Hence

$$\begin{aligned} D|_{A \times_T B}(a, b) &= \lim z_\alpha \cdot (a, b) - (a, b) \cdot z_\alpha + y_0 \cdot (a, b) - (a, b) \cdot y_0 \\ &= \lim(z_\alpha + y_0) \cdot (a, b) - (a, b) \cdot (z_\alpha + y_0). \end{aligned}$$

Therefore  $D|_{A \times_T B}$  is approximately inner. This completes the proof. □

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